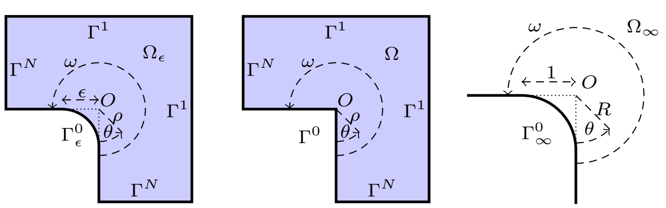
2D electrostatic problems with rounded corners

F. Buret*, M. Dauge¶, P. Dular†, L. Krähenbühl*, V. Péron‡, R. Perrussel*, C. Poignard§, and D. Voyer*
*Laboratoire Ampère CNRS UMR5005, Université de Lyon, École Centrale de Lyon, Écully, France
¶IRMAR CNRS UMR6625, Université de Rennes 1, Rennes, France
†F.R.S.-FNRS, ACE research-unit, Université de Liège, Liège, Belgium
‡LMAP CNRS UMR5142, INRIA Bordeaux-Sud-Ouest, Team MAGIQUE-3D, Pau, France
§INRIA Bordeaux-Sud-Ouest, Team MC2, IMB CNRS UMR5251, Université Bordeaux 1, Talence, France
mail to: laurent.krahenbuhl@ec-lyon.fr

***Abstract*—The second order terms of a multiscale expansion for dealing with rounded corners in 2D electrostatic problems are studied. The heuristics and the sequence of problems to be solved are presented and finite element simulations demonstrate the accuracy of the method.**

I. Introduction

High-voltage applications require a precise knowledge of the electric field in the area where the geometry of the structure is sharp. On a real device, the geometry is not "exactly sharp" but present rounded edges or corners. The accurate description of these rounded shapes, especially when the geometry involves several corners, can be cumbersome in a numerical model. In addition most of the time only a rough (statistical) description of the rounded shape is available due to the manufacturing tolerance. Dauge *et al.* have proposed in [1] a theoretical approach to tackle the "rounded shape" problems by an accurate asymptotic analysis.



(a) Domain with a rounded corner Ω_ϵ. (b) Domain with a sharp corner Ω. (c) Unbounded profile domain Ω_∞.

Fig. 1. Considered domains Ω_ϵ, Ω and Ω_∞.

Define the potentials v_ϵ in Ω_ϵ and v_0 in Ω (see Fig. 1) by

$$\begin{cases} \triangle v_\epsilon = 0, & \text{on } \Omega_\epsilon \\ v_\epsilon = 0, & \text{on } \Gamma^0_\epsilon \\ v_\epsilon = 1, & \text{on } \Gamma^1 \\ \partial_n v_\epsilon = 0, & \text{on } \Gamma^N \end{cases}, \quad \begin{cases} \triangle v_0 = 0, & \text{on } \Omega \\ v_0 = 0, & \text{on } \Gamma^0 \\ v_0 = 1, & \text{on } \Gamma^1 \\ \partial_n v_0 = 0, & \text{on } \Gamma^N \end{cases}, \tag{1}$$

where ϵ characterizes the "size" of the rounded corner and ∂_n denotes $n \cdot \nabla$, n being the unitary outward normal on the boundary of the domain. Throughout the paper ω denotes the angle of the sharp corner and $\alpha = \pi/\omega$. The main idea of [1] consists in expanding v_ϵ into two sums in power of ϵ^α. Using a smooth radial cut-off function φ defined by

$$\varphi(\rho) = \begin{cases} 1, & \text{if } \rho \geqslant d_1 \\ 0, & \text{if } \rho \leqslant d_0 \end{cases}, \quad \text{with } d_0 < d_1, \tag{2}$$

d_0, d_1 being fixed corner distances, the expansion writes for any integer $n \geqslant 1$ ($\varphi(./\epsilon)$ is the function $t \mapsto \varphi(t/\epsilon)$)

$$\begin{aligned} v_\epsilon =& \varphi\left(\frac{\cdot}{\epsilon}\right) v_0 + \varphi\left(\frac{\cdot}{\epsilon}\right) \sum_{p=2}^{n} b_p \epsilon^{p\alpha} v_{p\alpha} \\ &+ (1-\varphi) \sum_{p=1}^{n} B_p \epsilon^{p\alpha} V_{p\alpha}\left(\frac{\cdot}{\epsilon}\right) + r^\epsilon_{n\alpha}, \end{aligned} \tag{3}$$

where b_p and B_p are real parameters and $r^\epsilon_{n\alpha}$ is such that

$$\exists \tilde{\epsilon} > 0,\ \exists C_n > 0,\ \forall \epsilon < \tilde{\epsilon},\ \sqrt{\int_{\Omega_\epsilon} \|\nabla r^\epsilon_{n\alpha}\|^2 dx} < C_n \epsilon^{(n+1)\alpha}, \tag{4}$$

i.e. the energy norm of the error converges as $\epsilon^{(n+1)\alpha}$ to 0, which is written $r^\epsilon_{n\alpha} = \mathcal{O}_{H^1}(\epsilon^{(n+1)\alpha})$. The functions $v_{p\alpha}$ satisfy a boundary value problem in the sharp domain Ω (see Fig. 1(b)), whereas the functions $V_{q\alpha}$ are the so-called profile terms that satisfy Poisson's equation in the infinite domain Ω_∞ of $\mathbb{R}^2$, which is a localization of the rounded corner (see Fig. 1(c)).

The paper [2] was devoted to provide numerically the first order terms of the theoretical expansion of [1]. The following remarks have been observed:

- the exact solutions *close to the corner*, computed for several values of the curvature radius ϵ, are quasi-similar, up to a "scaling factor" (related to ϵ). It is also noticed that the "shape" of the solutions *close to the corner* (their "shape" but not their amplitude) weakly depend on other elements of the studied structure, such as the distance to the boundaries: if the corner geometry is *self-similar*[1], it is also said that the dominant term of the solutions close to the corner is *self-similar*.
- the exact solutions *far from the corner* are weakly influenced by the change of the curvature radius ϵ, and they converge to the solution on the domain with the sharp corner when ϵ goes to zero.

The aim of the present paper is to push forward numerically the expansion (3) on the numerical model studied in [2], especially when non-symmetric structures are involved.

[1]Roughly, it means that a single parameter, here ϵ, and a basic geometry are sufficient to describe the corners for any value of ϵ. For a precise definition, refer to [1].

II. Second order expansion

The heuristics for the construction of the two first orders are detailed in [1, subsection 4.1]. The roughest approximation v_0 of v_ϵ, far from the corner, that is defined in Ω by (1), writes in the neighborhood of the corner

$$v_0(x) \underset{\rho \to 0}{\simeq} \sum_{p=1}^{\infty} a_p \rho^{p\alpha} \sin(p\alpha\theta) = \sum_{p=1}^{\infty} a_p \mathfrak{s}^{p\alpha}(\rho, \theta), \qquad (5)$$

where (ρ, θ) are the polar coordinates. As the behaviors of v_0 and v_ϵ are different in the corner, v_0 should be truncated in the corner. Expansion (5) enforces the coefficient B_1 of (3) to equal the first singular coefficient a_1, while the profile term V_α satisfies, for $p = 1$,

$$\begin{cases} -\triangle_X V_{p\alpha} = [\triangle_X; \varphi] \mathfrak{s}^{p\alpha}, \text{ in } \Omega_\infty, \\ V_{p\alpha}|_{\Gamma^0_\infty} = 0, \\ \lim\limits_{R \to +\infty} V_{p\alpha} = 0. \end{cases} \qquad (6)$$

For any couple (ν, u), we remind that $[\Delta; \nu]u = \Delta(\nu u) - \nu\Delta u$. In [1], it is especially shown that in the neighborhood of $+\infty$

$$V_\alpha(X) \underset{\rho \to +\infty}{\simeq} \sum_{p=1}^{+\infty} A_p \mathfrak{s}^{-p\alpha}(X). \qquad (7)$$

Then, necessarily, b_2 and B_2 respectively equal $a_1 A_1$ and a_2 given by (5) and (7), leading to the second order:

$$\begin{aligned} v_\epsilon &= \varphi\left(\frac{\cdot}{\epsilon}\right) v_0 + \epsilon^\alpha (1-\varphi) a_1 V_\alpha \left(\frac{\cdot}{\epsilon}\right) \\ &+ \epsilon^{2\alpha} \left[a_1 A_1 v_{2\alpha} + (1-\varphi) a_2 V_{2\alpha} \left(\frac{\cdot}{\epsilon}\right) \right] + \mathcal{O}_{H^1}(\epsilon^{3\alpha}), \end{aligned} \qquad (8)$$

where $V_{2\alpha}$ is the profile term that satisfies (6) for $p = 2$ and $v_{2\alpha}$ is the correction far from the corner defined by

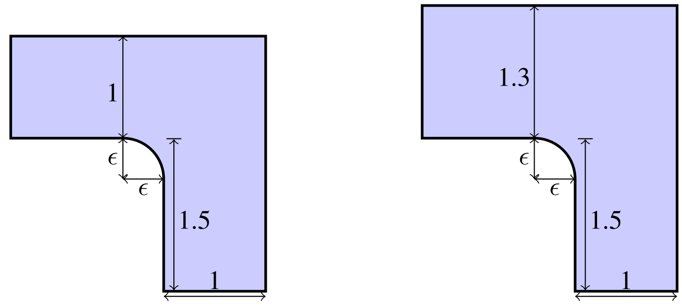
$$\begin{cases} -\triangle v_{2\alpha} = [\triangle; (1-\varphi)] \mathfrak{s}^{-\alpha}, \text{ on } \Omega, \\ v_{2\alpha} = 0, \text{ on } \Gamma^0, \text{ and } v_{2\alpha} = 0, \text{ on } \Gamma^1, \\ \partial_n v_{2\alpha} = 0, \text{ on } \Gamma^N. \end{cases} \qquad (9)$$

In [2, page 9], the profile problem (25) satisfied by v_∞ did not involve the cut-off function φ as in (6). Actually, the profiles V_α and v_∞ can be linked through the equality $V_\alpha = v_\infty - \varphi \mathfrak{s}^\alpha$. However, for profiles of order $p > 2$, the relationship is more complex and the use of the cut-off function seems to be more convenient [1].

III. Numerical results

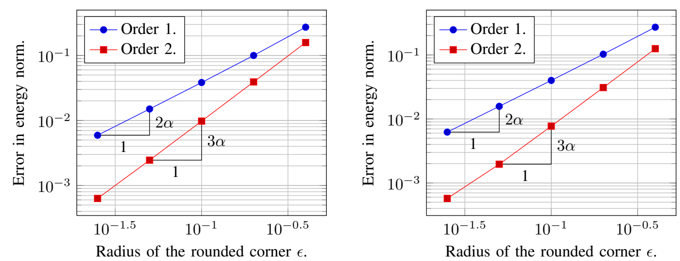
Two L-shape geometries with a rounded corner are considered; their dimensions are specified in Fig. 2. For both cases, α equals $2/3$. The finite element method has been implemented for solving (1), (6), and (9) similarly to [2].

The errors in the energy norm of the first and second order expansions, plotted in Fig. 3, behave respectively as $\epsilon^{2\alpha}$ and $\epsilon^{3\alpha}$, independently of the geometry. This is noteworthy since $a_2 = 0$ in the symmetric configuration (see Fig. 2(a)) and we could think that the first order terms would provide a better approximation than the first order terms in the non-symmetric configuration (see Fig. 2(b)). Nonetheless, this intuition is obviously not correct and the correction $v_{2\alpha}$ far from the corner plays an equivalent role than the correction close to the corner regarding the energy norm in the whole domain.



(a) Symmetric geometry. (b) Non-symmetric geometry.

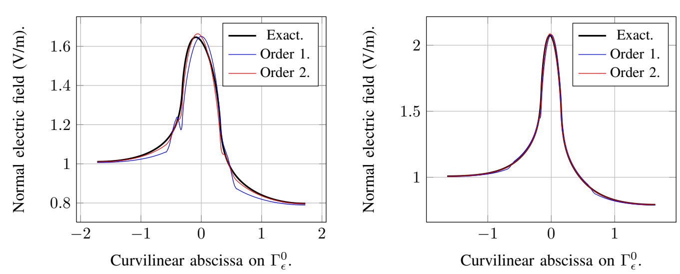
Fig. 2. Two considered problems and their dimensions. $\epsilon = 0.4$.



(a) Symmetric geometry. (b) Non-symmetric geometry.

Fig. 3. Convergence of the error in the energy norm.

The normal electric field along the electrodes in the non-symmetric configuration (from the bottom right to the top left of the electrode, see Fig. 2(b)) for two values of ϵ are presented in Fig. 4. This normal field has been computed respectively from v_ϵ, from the first order expansion and from the second order expansion given by (8). The behavior of the normal field is "closer" for the order 2, in particular the location of the maximum is roughly equivalent to the correct solution. However the maximum electric field is overestimated by both approximations, requiring to go further in the expansion.



(a) Rounded corner with $\epsilon = 0.4$. (b) Rounded corner with $\epsilon = 0.2$.

Fig. 4. Normal electric field along Γ^0_ϵ for the exact solution and the two first order approximations. Non-symmetric configuration is considered (Fig. 2(b)). Origin for the curvilinear abscissa is in the middle of the rounded corner.

References

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